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An overdetermined problem for harmonic functions

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ABSTRACT

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with connected real analytic boundary. We show that, if there exist n harmonic functions satisfying some appropriate boundary conditions, then Ω is a ball.

1. Introduction

The purpose of this paper is to prove the following theorem.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a bounded connected open set with connected real analytic boundary $\partial\Omega$. We denote by $\nu = (\nu_1, \ldots, \nu_n)$ the outward unit normal to $\partial\Omega$ and by H the mean curvature of $\partial\Omega$. Suppose that there exist a constant $c \in \mathbb{R}\backslash\{0\}$ and n harmonic functions in Ω $h_1, \ldots, h_n$ satisfying*

$$h_j = c\nu_j \quad \text{on } \partial\Omega, \tag{1}$$

and

$$\frac{\partial h_j}{\partial \nu} = -\big(1 + c(n-1)H\big)\nu_j \quad \text{on } \partial\Omega, \tag{2}$$

for $j = 1, \ldots, n$. Then Ω is a ball.

Notice that assumptions (1) and (2) imply that both $h = (h_1, \ldots, h_n)$ and $\partial h/\partial\nu = (\partial h_1/\partial\nu, \ldots, \partial h_n/\partial\nu)$ are parallel to ν.

Remark 1. Clearly, for each $j \in \{1, \ldots, n\}$, h_j extends to an analytic function on a neighborhood of $\partial\Omega$.

As in [1] we have the following corollary which gives a characterization of open balls in $\mathbb{R}^n$ by means of integral identities involving harmonic functions.

Corollary. *Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a bounded connected open set with connected real analytic boundary $\partial\Omega$. We denote by $\nu = (\nu_1, \ldots, \nu_n)$ the outward unit normal to $\partial\Omega$ and by H the mean curvature of $\partial\Omega$. If*

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$$C \int_{\partial\Omega} \frac{\partial h}{\partial \nu} \nu_j \, ds = -\int_{\partial\Omega} \big(1 + C(n-1)H\big) h \nu_j \, ds, \quad \forall j \in \{1, \dots, n\}, \tag{3}$$

for some constant $C \in \mathbb{R}$ and for every $h \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $\Delta h = 0$ in Ω, then Ω is a ball.

Remark 2. Notice that necessarily $C \neq 0$ in (3). Indeed let h be such that $\Delta h = 0$ in Ω and $h = \nu_j$ on $\partial\Omega$ for some $j \in \{1, \dots, n\}$. Then (3) implies that $C \neq 0$.

Remark 3. Suppose that there exist a constant $c \in \mathbb{R}\backslash\{0\}$ and n harmonic functions $h_1, \dots, h_n$ satisfying (1)–(2). Using Green's identity we get

$$\int_{\partial\Omega} h_j \frac{\partial h}{\partial \nu} \, ds = \int_{\partial\Omega} h \frac{\partial h_j}{\partial \nu} \, ds, \quad j = 1, \dots, n,$$

for every harmonic function $h \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then (3) holds with $C = c$.

We shall need the following result [8].

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ ($n \geqslant 2$) be a bounded connected open set with C^2 boundary. Suppose that there exists $u \in C^2(\overline{\Omega})$ such that*

$$\Delta u + 1 = 0 \quad \text{in } \Omega,$$
$$u = 0, \qquad \frac{\partial u}{\partial \nu} = d \quad \text{on } \partial\Omega,$$

where d denotes some constant. Then Ω is a ball of center x_0 and radius $R = -dn$ and $u(x) = (R^2 - |x - x_0|^2)/2n$ for $x \in \Omega$.

Remark 4. Let u be as in Theorem 2. For each $j \in \{1, \dots, n\}$, $\partial u/\partial x_j$ is harmonic in Ω. Since $u = 0$ on $\partial\Omega$, we have $\partial u/\partial x_j = (\partial u/\partial \nu)\nu_j = d\nu_j$ on $\partial\Omega$ for $j = 1, \dots, n$. Now, arguing as in the proof of Lemma 2 below, we easily obtain

$$\frac{\partial}{\partial \nu}\left(\frac{\partial u}{\partial x_j}\right) = -\big(1 + d(n-1)H\big)\nu_j, \quad j = 1, \dots, n.$$

To the best of our knowledge Theorem 1 is the first result in this direction. It would be interesting to give a proof of Theorem 1 that does not make use of Serrin's result. Finally let us also mention that there is a huge amount of literature concerning overdetermined problems in both bounded and unbounded domains: See [2–7,9–11] and the references therein.

2. Proof of Theorem 1

$\partial\Omega$ is locally the graph of an analytic function of $n-1$ variables φ, that is, $\partial\Omega$ is locally defined by $x_n = \varphi(x_1, \dots, x_{n-1})$ and $x = (x_1, \dots, x_n) = (x', x_n) \in \Omega$ if and only if $x_n < \varphi(x_1, \dots, x_{n-1}) = \varphi(x')$. Then the outward unit normal ν and the mean curvature H at $x = (x', \varphi(x')) \in \partial\Omega$ are given by

$$\nu(x) = \frac{1}{\sqrt{1 + |\nabla\varphi(x')|^2}}\big(-\nabla\varphi(x'), 1\big),$$

and

$$H(x) = \frac{1}{n-1} \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j}\big(\nu_j(x)\big).$$

We shall use repeatedly the following lemma.

Lemma 1. *We have*

$$\sum_{j=1}^{n-1} \nu_j \frac{\partial}{\partial x_j}(\nu_n) = \nu_n\big((n-1)H + \nu_n \Delta\varphi\big),$$

and

$$\sum_{j=1}^{n-1} \nu_j \frac{\partial}{\partial x_j}(\nu_p) = -\frac{1}{\nu_n} \frac{\partial}{\partial x_p}(\nu_n) + \nu_p\big((n-1)H + \nu_n \Delta\varphi\big)$$

for $1 \leqslant p \leqslant n-1$.

Proof. Let $p \in \{1, \dots, n-1\}$. We have

$$\begin{aligned}
\sum_{j=1}^{n-1} \nu_j \frac{\partial}{\partial x_j}(\nu_p) &= \sum_{j=1}^{n-1} \nu_j \left(-\nu_n \frac{\partial^2 \varphi}{\partial x_j \partial x_p} + \nu_n^3 \frac{\partial \varphi}{\partial x_p} \sum_{k=1}^{n-1} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \frac{\partial \varphi}{\partial x_k} \right) \\
&= \nu_n^2 \sum_{j=1}^{n-1} \frac{\partial^2 \varphi}{\partial x_j \partial x_p} \frac{\partial \varphi}{\partial x_j} + \nu_p \nu_n^3 \sum_{k=1}^{n-1} \frac{\partial \varphi}{\partial x_k} \left(\sum_{j=1}^{n-1} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \frac{\partial \varphi}{\partial x_j} \right) \\
&= -\frac{1}{\nu_n} \frac{\partial}{\partial x_p}(\nu_n) + \nu_p \sum_{k=1}^{n-1} \left(\frac{\partial}{\partial x_k}(\nu_k) + \nu_n \frac{\partial^2 \varphi}{\partial x_k^2} \right) \\
&= -\frac{1}{\nu_n} \frac{\partial}{\partial x_p}(\nu_n) + \nu_p \big((n-1)H + \nu_n \Delta\varphi \big).
\end{aligned}$$

The proof is analogous when $p = n$. $\square$

In view of Remark 4, our first aim is to show that there exists an analytic function w satisfying

$$\frac{\partial w}{\partial x_j} = h_j \quad \text{in } \Omega, \ j = 1, \dots, n.$$

Therefore it is natural to expect that the functions h_j satisfy Lemma 5 below. Unfortunately long and tedious calculations are required.

Lemma 2. *We have*

$$\frac{\partial h_j}{\partial x_k} = -\big(1 + 2c(n-1)H + c\nu_n \Delta\varphi\big)\nu_j \nu_k + c\frac{\nu_k}{\nu_n} \frac{\partial}{\partial x_j}(\nu_n) + c\frac{\partial}{\partial x_k}(\nu_j)$$

on $\partial\Omega$ for $1 \leqslant j, k \leqslant n$. Moreover

$$\frac{\partial h_j}{\partial x_k} = \frac{\partial h_k}{\partial x_j} \quad \text{for } 1 \leqslant j, k \leqslant n, \quad \text{and} \quad \sum_{j=1}^{n} \frac{\partial h_j}{\partial x_j} = -1$$

on $\partial\Omega$.

Proof. Differentiating (1) with respect to x_k for $k \in \{1, \dots, n-1\}$ and multiplying by ν_n we get

$$\nu_n \frac{\partial h_j}{\partial x_k} - \nu_k \frac{\partial h_j}{\partial x_n} = c\nu_n \frac{\partial}{\partial x_k}(\nu_j), \tag{4}$$

for $1 \leqslant j \leqslant n$. Multiplying (4) by ν_k and adding we obtain

$$\nu_n \frac{\partial h_j}{\partial \nu} - \frac{\partial h_j}{\partial x_n} = c\nu_n \sum_{k=1}^{n-1} \nu_k \frac{\partial}{\partial x_k}(\nu_j).$$

Then using (2) we deduce that

$$\frac{\partial h_j}{\partial x_n} = -\big(1 + c(n-1)H\big)\nu_j \nu_n - c\nu_n \sum_{k=1}^{n-1} \nu_k \frac{\partial}{\partial x_k}(\nu_j), \tag{5}$$

for $1 \leqslant j \leqslant n$. Now (4) and (5) imply that

$$\frac{\partial h_j}{\partial x_k} = -\big(1 + c(n-1)H\big)\nu_j \nu_k - c\nu_k \sum_{r=1}^{n-1} \nu_r \frac{\partial}{\partial x_r}(\nu_j) + c\frac{\partial}{\partial x_k}(\nu_j), \tag{6}$$

for $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant n-1$. (5), (6) and Lemma 1 give the first part of the lemma. Now with the help of Lemma 1 we get

$$\sum_{j=1}^{n} \frac{\partial h_j}{\partial x_j} = -\bigl(1 + 2c(n-1)H + c\nu_n \Delta\varphi\bigr) + \frac{c}{\nu_n} \sum_{j=1}^{n-1} \nu_j \frac{\partial}{\partial x_j}(\nu_n) + c \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j}(\nu_j)$$
$$= -\bigl(1 + 2c(n-1)H + c\nu_n \Delta\varphi\bigr) + c\bigl((n-1)H + \nu_n \Delta\varphi\bigr) + c(n-1)H$$
$$= -1$$

on $\partial\Omega$. Finally for $1 \leqslant j, k \leqslant n-1$ we have

$$\frac{\nu_k}{\nu_n} \frac{\partial}{\partial x_j}(\nu_n) + \frac{\partial}{\partial x_k}(\nu_j) = -\nu_k \nu_n^2 \sum_{i=1}^{n-1} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \frac{\partial \varphi}{\partial x_i} - \nu_n \frac{\partial^2 \varphi}{\partial x_j \partial x_k} + \nu_n^3 \frac{\partial \varphi}{\partial x_j} \sum_{i=1}^{n-1} \frac{\partial^2 \partial \varphi}{\partial x_i \partial x_k} \frac{\partial \varphi}{\partial x_i}$$
$$= -\nu_k \nu_n^2 \sum_{i=1}^{n-1} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \frac{\partial \varphi}{\partial x_i} - \nu_n \frac{\partial^2 \varphi}{\partial x_j \partial x_k} - \nu_j \nu_n^2 \sum_{i=1}^{n-1} \frac{\partial^2 \partial \varphi}{\partial x_i \partial x_k} \frac{\partial \varphi}{\partial x_i}$$
$$= \frac{\nu_j}{\nu_n} \frac{\partial}{\partial x_k}(\nu_n) + \frac{\partial}{\partial x_j}(\nu_k),$$

and we conclude with the first part of the lemma. □

Lemma 3. *Define $a_{jk} : \partial\Omega \to \mathbb{R}$ by $a_{jk} = \partial h_j / \partial x_k$. We have*

$$\nu_n \frac{\partial^2 h_j}{\partial x_k \partial x_p} = -\nu_k \nu_p \left(\nu_n \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}(a_{ij}) + \sum_{i=1}^{n-1} \nu_i \frac{\partial}{\partial x_i}(a_{jn}) \right) + \nu_n \frac{\partial}{\partial x_p}(a_{jk}) + \nu_p \frac{\partial}{\partial x_k}(a_{jn})$$

on $\partial\Omega$ for $1 \leqslant j, k, p \leqslant n$.

Proof. Differentiating a_{jk} with respect to x_p for $p \in \{1, \ldots, n-1\}$ and multiplying by ν_n we get

$$\nu_n \frac{\partial^2 h_j}{\partial x_k \partial x_p} - \nu_p \frac{\partial^2 h_j}{\partial x_k \partial x_n} = \nu_n \frac{\partial}{\partial x_p}(a_{jk}), \tag{7}$$

for $1 \leqslant j, k \leqslant n$. Let $k = p$ in (7). Adding and using Lemma 2 we obtain

$$\frac{\partial}{\partial \nu}\left(\frac{\partial h_j}{\partial x_n}\right) = \nu_n \Delta h_j - \nu_n \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}(a_{ij}) = -\nu_n \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}(a_{ij}), \tag{8}$$

for $1 \leqslant j \leqslant n$. Multiplying (7) by ν_p and adding we obtain

$$\frac{\partial^2 h_j}{\partial x_k \partial x_n} = \nu_n \frac{\partial}{\partial \nu}\left(\frac{\partial h_j}{\partial x_k}\right) - \nu_n \sum_{i=1}^{n-1} \nu_i \frac{\partial}{\partial x_i}(a_{jk}), \tag{9}$$

for $1 \leqslant j, k \leqslant n$. From (8) and (9) we deduce that

$$\frac{\partial^2 h_j}{\partial x_n^2} = -\nu_n \left(\nu_n \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}(a_{ij}) + \sum_{i=1}^{n-1} \nu_i \frac{\partial}{\partial x_i}(a_{jn}) \right) \tag{10}$$

for $1 \leqslant j \leqslant n$. From (7) and (10) we get

$$\nu_n \frac{\partial^2 h_j}{\partial x_p \partial x_n} = \nu_p \frac{\partial^2 h_j}{\partial x_n^2} + \nu_n \frac{\partial}{\partial x_p}(a_{jn})$$
$$= -\nu_p \nu_n \left(\nu_n \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}(a_{ij}) + \sum_{i=1}^{n-1} \nu_i \frac{\partial}{\partial x_i}(a_{jn}) \right) + \nu_n \frac{\partial}{\partial x_p}(a_{jn}),$$

for $1 \leqslant j \leqslant n$ and $1 \leqslant p \leqslant n-1$, hence

$$\frac{\partial^2 h_j}{\partial x_p \partial x_n} = -\nu_p \left(\nu_n \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}(a_{ij}) + \sum_{i=1}^{n-1} \nu_i \frac{\partial}{\partial x_i}(a_{jn}) \right) + \frac{\partial}{\partial x_p}(a_{jn}), \tag{11}$$

for $1 \leqslant j \leqslant n$ and $1 \leqslant p \leqslant n-1$. Finally using (7) and (11) we have

$$\begin{aligned}\nu_n \frac{\partial^2 h_j}{\partial x_k \partial x_p} &= \nu_p \frac{\partial^2 h_j}{\partial x_k \partial x_n} + \nu_n \frac{\partial}{\partial x_p}(a_{jk}) \\ &= -\nu_k \nu_p \left(\nu_n \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}(a_{ij}) + \sum_{i=1}^{n-1} \nu_i \frac{\partial}{\partial x_i}(a_{jn}) \right) + \nu_p \frac{\partial}{\partial x_k}(a_{jn}) + \nu_n \frac{\partial}{\partial x_p}(a_{jk}),\end{aligned}$$

for $1 \leqslant j \leqslant n$ and $1 \leqslant k, p \leqslant n-1$ and the lemma is proved. □

Lemma 4. *We have*

$$\frac{\partial^2 h_j}{\partial x_k \partial x_p} = \frac{\partial^2 h_k}{\partial x_j \partial x_p}$$

on $\partial\Omega$ for $1 \leqslant j, k, p \leqslant n$.

Proof. It is enough to show that

$$\frac{\partial^2 h_j}{\partial x_n^2} = \frac{\partial^2 h_n}{\partial x_j \partial x_n} \tag{12}$$

on $\partial\Omega$ for $1 \leqslant j \leqslant n-1$. Indeed assume that (12) holds. (7) and Lemma 2 imply that

$$\begin{aligned}\nu_n \frac{\partial^2 h_j}{\partial x_p \partial x_n} &= \nu_p \frac{\partial^2 h_j}{\partial x_n^2} + \nu_n \frac{\partial}{\partial x_p}(a_{jn}) \\ &= \nu_p \frac{\partial^2 h_n}{\partial x_j \partial x_n} + \nu_n \frac{\partial}{\partial x_p}(a_{jn}) \\ &= \nu_n \frac{\partial^2 h_n}{\partial x_p \partial x_j},\end{aligned}$$

for $1 \leqslant j \leqslant n$ and $1 \leqslant p \leqslant n-1$. Therefore the lemma holds for $k = n$ and $1 \leqslant j, p \leqslant n$. Then we have

$$\frac{\partial^2 h_j}{\partial x_k \partial x_n} = \frac{\partial^2 h_n}{\partial x_j \partial x_k} = \frac{\partial^2 h_k}{\partial x_j \partial x_n}.$$

Using (7) again we get

$$\begin{aligned}\nu_n \frac{\partial^2 h_j}{\partial x_k \partial x_p} &= \nu_p \frac{\partial^2 h_j}{\partial x_k \partial x_n} + \nu_n \frac{\partial}{\partial x_p}(a_{jk}) = \nu_p \frac{\partial^2 h_n}{\partial x_j \partial x_k} + \nu_n \frac{\partial}{\partial x_p}(a_{jk}) \\ &= \nu_p \frac{\partial^2 h_k}{\partial x_j \partial x_n} + \nu_n \frac{\partial}{\partial x_p}(a_{jk}) = \nu_n \frac{\partial^2 h_k}{\partial x_j \partial x_p},\end{aligned}$$

and the lemma is proved. Now we prove (12). In view of Lemma 3 we have to show that

$$\nu_n \left(\nu_n \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}(a_{ij}) + \sum_{i=1}^{n-1} \nu_i \frac{\partial}{\partial x_i}(a_{jn}) \right) = \nu_j \left(\nu_n \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}(a_{in}) + \sum_{i=1}^{n-1} \nu_i \frac{\partial}{\partial x_i}(a_{nn}) \right) - \frac{\partial}{\partial x_j}(a_{nn}), \tag{13}$$

on $\partial\Omega$ for $1 \leqslant j \leqslant n-1$. Define

$$a = 1 + 2c(n-1)H + c\nu_n \Delta\varphi.$$

By Lemma 2 we have

$$a_{ij} = a_{ji} = -a\nu_i \nu_j + c\frac{\nu_i}{\nu_n} \frac{\partial}{\partial x_j}(\nu_n) + c\frac{\partial}{\partial x_i}(\nu_j).$$

Then, using Lemma 1, we get

$$\begin{aligned}\sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}(a_{ij}) &= -\nu_j \sum_{i=1}^{n-1} \nu_i \frac{\partial}{\partial x_i}(a) - a \sum_{i=1}^{n-1} \left(\nu_j \frac{\partial}{\partial x_i}(\nu_i) + \nu_i \frac{\partial}{\partial x_i}(\nu_j) \right) \\ &\quad - c\Delta\varphi \frac{\partial}{\partial x_j}(\nu_n) + \frac{c}{\nu_n} \sum_{i=1}^{n-1} \nu_i \frac{\partial^2}{\partial x_i \partial x_j}(\nu_n) + c\sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2}(\nu_j)\end{aligned}$$

$$= -\nu_j \sum_{i=1}^{n-1} \nu_i \frac{\partial}{\partial x_i}(a) - a\left(2(n-1)H\nu_j - \frac{1}{\nu_n}\frac{\partial}{\partial x_j}(\nu_n) + \nu_n\nu_j\Delta\varphi\right)$$
$$- c\Delta\varphi \frac{\partial}{\partial x_j}(\nu_n) + \frac{c}{\nu_n}\sum_{i=1}^{n-1}\nu_i \frac{\partial^2}{\partial x_i \partial x_j}(\nu_n) + c\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}(\nu_j),$$

and

$$\sum_{i=1}^{n-1}\nu_i \frac{\partial}{\partial x_i}(a_{jn}) = -\nu_j\nu_n \sum_{i=1}^{n-1}\nu_i \frac{\partial}{\partial x_i}(a) - a\sum_{i=1}^{n-1}\nu_i\left(\nu_j \frac{\partial}{\partial x_i}(\nu_n) + \nu_n \frac{\partial}{\partial x_i}(\nu_j)\right) + c\sum_{i=1}^{n-1}\nu_i \frac{\partial^2}{\partial x_i \partial x_j}(\nu_n)$$
$$= -\nu_j\nu_n \sum_{i=1}^{n-1}\nu_i \frac{\partial}{\partial x_i}(a) + c\sum_{i=1}^{n-1}\nu_i \frac{\partial^2}{\partial x_i \partial x_j}(\nu_n) - a\left(2\nu_j\nu_n\big((n-1)H + \nu_n\Delta\varphi\big) - \frac{\partial}{\partial x_j}(\nu_n)\right),$$

from which we deduce that

$$\nu_n\nu_j \sum_{i=1}^{n-1}\frac{\partial}{\partial x_i}(a_{in}) - \nu_n^2 \sum_{i=1}^{n-1}\frac{\partial}{\partial x_i}(a_{ij}) = -a\nu_n \frac{\partial}{\partial x_j}(\nu_n) + c\nu_n^2\Delta\varphi \frac{\partial}{\partial x_j}(\nu_n) - c\nu_n \sum_{i=1}^{n-1}\nu_i \frac{\partial^2}{\partial x_i \partial x_j}(\nu_n)$$
$$- c\nu_n^2 \sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}(\nu_j) + c\nu_n\nu_j \sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}(\nu_n),$$

and

$$\nu_j \sum_{i=1}^{n-1}\nu_i \frac{\partial}{\partial x_i}(a_{nn}) - \nu_n \sum_{i=1}^{n-1}\nu_i \frac{\partial}{\partial x_i}(a_{jn}) = -a\nu_n \frac{\partial}{\partial x_j}(\nu_n) - c\nu_n \sum_{i=1}^{n-1}\nu_i \frac{\partial^2}{\partial x_i \partial x_j}(\nu_n).$$

Define

$$A = c\nu_n^2\Delta\varphi \frac{\partial}{\partial x_j}(\nu_n) + \nu_n^2 \frac{\partial}{\partial x_j}(a) - 2c\nu_n \sum_{i=1}^{n-1}\nu_i \frac{\partial^2}{\partial x_i \partial x_j}(\nu_n) - c\nu_n^2 \sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}(\nu_j) + c\nu_n\nu_j \sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}(\nu_n).$$

Since

$$\frac{\partial}{\partial x_j}(a_{nn}) = -a\frac{\partial}{\partial x_j}\big(\nu_n^2\big) - \nu_n^2 \frac{\partial}{\partial x_j}(a),$$

we deduce that (13) is equivalent to $A = 0$. Let $1 \leqslant i, j \leqslant n-1$. We have

$$\frac{\partial}{\partial x_i}(\nu_n) = -\nu_n^3 \sum_{k=1}^{n-1}\frac{\partial^2\varphi}{\partial x_i \partial x_k}\frac{\partial\varphi}{\partial x_k},$$

and

$$\frac{\partial}{\partial x_i}(\nu_j) = -\nu_n \frac{\partial^2\varphi}{\partial x_i \partial x_j} + \nu_n^3 \sum_{k=1}^{n-1}\frac{\partial^2\varphi}{\partial x_i \partial x_k}\frac{\partial\varphi}{\partial x_k}\frac{\partial\varphi}{\partial x_j}.$$

Then

$$\frac{\partial^2}{\partial x_i^2}(\nu_n) = \frac{3}{\nu_n}\left(\frac{\partial}{\partial x_i}(\nu_n)\right)^2 - \nu_n^3 \sum_{k=1}^{n-1}\left(\frac{\partial^3\varphi}{\partial x_i^2 \partial x_k}\frac{\partial\varphi}{\partial x_k} + \left(\frac{\partial^2\varphi}{\partial x_i \partial x_k}\right)^2\right),$$
$$\frac{\partial^2}{\partial x_i \partial x_j}(\nu_n) = \frac{3}{\nu_n}\frac{\partial}{\partial x_i}(\nu_n)\frac{\partial}{\partial x_j}(\nu_n) - \nu_n^3 \sum_{k=1}^{n-1}\left(\frac{\partial^3\varphi}{\partial x_i \partial x_j \partial x_k}\frac{\partial\varphi}{\partial x_k} + \frac{\partial^2\varphi}{\partial x_i \partial x_k}\frac{\partial^2\varphi}{\partial x_j \partial x_k}\right)$$

and

$$\frac{\partial^2}{\partial x_i^2}(\nu_j) = -\frac{\partial^2\varphi}{\partial x_i \partial x_j}\frac{\partial}{\partial x_i}(\nu_n) - \nu_n \frac{\partial^3\varphi}{\partial x_i^2 \partial x_j} + \frac{3}{\nu_n^2}\nu_j\left(\frac{\partial}{\partial x_i}(\nu_n)\right)^2$$
$$+ \nu_n^3 \sum_{k=1}^{n-1}\left(\left(\frac{\partial^3\varphi}{\partial x_i^2 \partial x_k}\frac{\partial\varphi}{\partial x_k} + \left(\frac{\partial^2\varphi}{\partial x_i \partial x_k}\right)^2\right)\frac{\partial\varphi}{\partial x_j} + \frac{\partial^2\varphi}{\partial x_i \partial x_k}\frac{\partial^2\varphi}{\partial x_i \partial x_j}\frac{\partial\varphi}{\partial x_k}\right).$$

We deduce that

$$\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}(\nu_n)=\frac{3}{\nu_n}\sum_{i=1}^{n-1}\left(\frac{\partial}{\partial x_i}(\nu_n)\right)^2+\nu_n^2\sum_{k=1}^{n-1}\nu_k\frac{\partial\Delta\varphi}{\partial x_k}-\nu_n^3\sum_{i,k=1}^{n-1}\left(\frac{\partial^2\varphi}{\partial x_i\partial x_k}\right)^2,$$

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}(\nu_j)={}&-2\sum_{i=1}^{n-1}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\frac{\partial}{\partial x_i}(\nu_n)-\nu_n\frac{\partial\Delta\varphi}{\partial x_j}+\frac{3}{\nu_n^2}\nu_j\sum_{i=1}^{n-1}\left(\frac{\partial}{\partial x_i}(\nu_n)\right)^2\\&+\nu_n\nu_j\sum_{k=1}^{n-1}\nu_k\frac{\partial\Delta\varphi}{\partial x_k}-\nu_n^2\nu_j\sum_{i,k=1}^{n-1}\left(\frac{\partial^2\varphi}{\partial x_i\partial x_k}\right)^2,\end{aligned}$$

and with the help of Lemma 1

$$\sum_{i=1}^{n-1}\nu_i\frac{\partial^2}{\partial x_i\partial x_j}(\nu_n)=3\big((n-1)H+\nu_n\Delta\varphi\big)\frac{\partial}{\partial x_j}(\nu_n)+\nu_n^2\sum_{i,k=1}^{n-1}\frac{\partial^3\varphi}{\partial x_i\partial x_j\partial x_k}\nu_i\nu_k-\nu_n\sum_{k=1}^{n-1}\frac{\partial^2\varphi}{\partial x_j\partial x_k}\frac{\partial}{\partial x_k}(\nu_n).$$

Therefore

$$\begin{aligned}A={}&-4c\nu_n^2\Delta\varphi\frac{\partial}{\partial x_j}(\nu_n)-6c\nu_n(n-1)H\frac{\partial}{\partial x_j}(\nu_n)-2c\nu_n^3\sum_{i,k=1}^{n-1}\frac{\partial^3\varphi}{\partial x_i\partial x_j\partial x_k}\nu_i\nu_k\\&+4c\nu_n^2\sum_{k=1}^{n-1}\frac{\partial^2\varphi}{\partial x_j\partial x_k}\frac{\partial}{\partial x_k}(\nu_n)+2c\nu_n^3\frac{\partial\Delta\varphi}{\partial x_j}+2c(n-1)\nu_n^2\frac{\partial}{\partial x_j}(H).\end{aligned}$$

Now we have

$$\begin{aligned}(n-1)H&=\sum_{i=1}^{n-1}\frac{\partial}{\partial x_i}(\nu_i)=\sum_{i=1}^{n-1}\left(-\frac{\partial^2\varphi}{\partial x_i^2}\nu_n+\nu_n^3\sum_{k=1}^{n-1}\frac{\partial^2\varphi}{\partial x_i\partial x_k}\frac{\partial\varphi}{\partial x_k}\frac{\partial\varphi}{\partial x_i}\right)\\&=-\nu_n\Delta\varphi+\nu_n^3\sum_{i,k=1}^{n-1}\frac{\partial^2\varphi}{\partial x_i\partial x_k}\frac{\partial\varphi}{\partial x_k}\frac{\partial\varphi}{\partial x_i},\end{aligned}$$

hence

$$\begin{aligned}(n-1)\frac{\partial}{\partial x_j}(H)={}&-\Delta\varphi\frac{\partial}{\partial x_j}(\nu_n)-\nu_n\frac{\partial\Delta\varphi}{\partial x_j}+3\nu_n^2\frac{\partial}{\partial x_j}(\nu_n)\sum_{i,k=1}^{n-1}\frac{\partial^2\varphi}{\partial x_i\partial x_k}\frac{\partial\varphi}{\partial x_k}\frac{\partial\varphi}{\partial x_i}\\&+\nu_n^3\sum_{i,k=1}^{n-1}\left(\frac{\partial^3\varphi}{\partial x_i\partial x_j\partial x_k}\frac{\partial\varphi}{\partial x_k}\frac{\partial\varphi}{\partial x_i}+\frac{\partial^2\varphi}{\partial x_i\partial x_k}\frac{\partial^2\varphi}{\partial x_j\partial x_k}\frac{\partial\varphi}{\partial x_i}+\frac{\partial^2\varphi}{\partial x_i\partial x_k}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\frac{\partial\varphi}{\partial x_k}\right)\\={}&-\Delta\varphi\frac{\partial}{\partial x_j}(\nu_n)-\nu_n\frac{\partial\Delta\varphi}{\partial x_j}-3\nu_n\frac{\partial}{\partial x_j}(\nu_n)\sum_{i,k=1}^{n-1}\nu_i\frac{\partial^2\varphi}{\partial x_i\partial x_k}\frac{\partial\varphi}{\partial x_k}\\&+\nu_n\sum_{i,k=1}^{n-1}\frac{\partial^3\varphi}{\partial x_i\partial x_j\partial x_k}\nu_i\nu_k-2\sum_{k=1}^{n-1}\frac{\partial^2\varphi}{\partial x_j\partial x_k}\frac{\partial}{\partial x_k}(\nu_n)\\={}&-\Delta\varphi\frac{\partial}{\partial x_j}(\nu_n)-\nu_n\frac{\partial\Delta\varphi}{\partial x_j}+\frac{3}{\nu_n^2}\frac{\partial}{\partial x_j}(\nu_n)\sum_{i=1}^{n-1}\nu_i\frac{\partial}{\partial x_i}(\nu_n)\\&+\nu_n\sum_{i,k=1}^{n-1}\frac{\partial^3\varphi}{\partial x_i\partial x_j\partial x_k}\nu_i\nu_k-2\sum_{k=1}^{n-1}\frac{\partial^2\varphi}{\partial x_j\partial x_k}\frac{\partial}{\partial x_k}(\nu_n)\\={}&2\Delta\varphi\frac{\partial}{\partial x_j}(\nu_n)-\nu_n\frac{\partial\Delta\varphi}{\partial x_j}+\frac{3}{\nu_n}(n-1)H\frac{\partial}{\partial x_j}(\nu_n)+\nu_n\sum_{i,k=1}^{n-1}\frac{\partial^3\varphi}{\partial x_i\partial x_j\partial x_k}\nu_i\nu_k-2\sum_{k=1}^{n-1}\frac{\partial^2\varphi}{\partial x_j\partial x_k}\frac{\partial}{\partial x_k}(\nu_n)\end{aligned}$$

where we have again used Lemma 1. Now we easily verify that $A=0$ and (13) is proved. The proof of the lemma is complete. $\square$

Lemma 5. *Let $\alpha \in \mathbb{N}^n$. We have*

$$\frac{\partial^{|\alpha|+1} h_j}{\partial x_k \partial x^\alpha} = \frac{\partial^{|\alpha|+1} h_k}{\partial x_j \partial x^\alpha} \tag{14}$$

on $\partial\Omega$ for $1 \leqslant j, k \leqslant n$.

Proof. Lemmas 3 and 4 imply that (14) holds for $|\alpha| \leqslant 1$. Now suppose that (14) holds for $|\alpha| = m \geqslant 1$. Then for all $\beta \in \mathbb{N}^n$ such that $|\beta| = m - 1$ and for all $p \in \{1, \dots, n\}$ we have

$$\frac{\partial^{|\beta|+2} h_j}{\partial x_k \partial x_p \partial x^\beta} = \frac{\partial^{|\beta|+2} h_k}{\partial x_j \partial x_p \partial x^\beta} \tag{15}$$

on $\partial\Omega$ for $1 \leqslant j, k \leqslant n$. Differentiating (15) with respect to x_i for $i \in \{1, \dots, n-1\}$ and multiplying by ν_n we get

$$\nu_n \frac{\partial^{|\beta|+3} h_j}{\partial x_i \partial x_k \partial x_p \partial x^\beta} - \nu_i \frac{\partial^{|\beta|+3} h_j}{\partial x_k \partial x_p \partial x_n \partial x^\beta} = \nu_n \frac{\partial^{|\beta|+3} h_k}{\partial x_i \partial x_j \partial x_p \partial x^\beta} - \nu_i \frac{\partial^{|\beta|+3} h_k}{\partial x_j \partial x_p \partial x_n \partial x^\beta}, \tag{16}$$

for $1 \leqslant j, k, p \leqslant n$. Let $i = p$ in (16). Adding we get

$$\nu_n \Delta \frac{\partial^{|\beta|+1} h_j}{\partial x_k \partial x^\beta} - \frac{\partial}{\partial \nu}\left(\frac{\partial^{|\beta|+2} h_j}{\partial x_k \partial x_n \partial x^\beta}\right) = \nu_n \Delta \frac{\partial^{|\beta|+1} h_k}{\partial x_j \partial x^\beta} - \frac{\partial}{\partial \nu}\left(\frac{\partial^{|\beta|+2} h_k}{\partial x_j \partial x_n \partial x^\beta}\right).$$

Therefore we have

$$\frac{\partial}{\partial \nu}\left(\frac{\partial^{|\beta|+2} h_j}{\partial x_k \partial x_n \partial x^\beta}\right) = \frac{\partial}{\partial \nu}\left(\frac{\partial^{|\beta|+2} h_k}{\partial x_j \partial x_n \partial x^\beta}\right), \tag{17}$$

for $1 \leqslant j, k \leqslant n$. Multiplying (16) by ν_i and adding we obtain

$$\nu_n \frac{\partial}{\partial \nu}\left(\frac{\partial^{|\beta|+2} h_j}{\partial x_k \partial x_p \partial x^\beta}\right) - \frac{\partial^{|\beta|+3} h_j}{\partial x_k \partial x_p \partial x_n \partial x^\beta} = \nu_n \frac{\partial}{\partial \nu}\left(\frac{\partial^{|\beta|+2} h_k}{\partial x_j \partial x_p \partial x^\beta}\right) - \frac{\partial^{|\beta|+3} h_k}{\partial x_j \partial x_p \partial x_n \partial x^\beta}, \tag{18}$$

for $1 \leqslant j, k, p \leqslant n$. Multiplying (16) by ν_p and adding we obtain

$$\nu_n \frac{\partial}{\partial \nu}\left(\frac{\partial^{|\beta|+2} h_j}{\partial x_i \partial x_k \partial x^\beta}\right) - \nu_i \frac{\partial}{\partial \nu}\left(\frac{\partial^{|\beta|+2} h_j}{\partial x_k \partial x_n \partial x^\beta}\right) = \nu_n \frac{\partial}{\partial \nu}\left(\frac{\partial^{|\beta|+2} h_k}{\partial x_i \partial x_j \partial x^\beta}\right) - \nu_i \frac{\partial}{\partial \nu}\left(\frac{\partial^{|\beta|+2} h_k}{\partial x_j \partial x_n \partial x^\beta}\right), \tag{19}$$

for $1 \leqslant j, k \leqslant n$ and $1 \leqslant i \leqslant n-1$. (17) and (19) imply that

$$\frac{\partial}{\partial \nu}\left(\frac{\partial^{|\beta|+2} h_j}{\partial x_i \partial x_k \partial x^\beta}\right) = \frac{\partial}{\partial \nu}\left(\frac{\partial^{|\beta|+2} h_k}{\partial x_i \partial x_j \partial x^\beta}\right), \tag{20}$$

for $1 \leqslant i \leqslant n-1$ and $1 \leqslant j, k \leqslant n$. Then (17), (18) and (20) give

$$\frac{\partial^{|\beta|+3} h_j}{\partial x_k \partial x_p \partial x_n \partial x^\beta} = \frac{\partial^{|\beta|+3} h_k}{\partial x_j \partial x_p \partial x_n \partial x^\beta}, \tag{21}$$

for $1 \leqslant j, k, p \leqslant n$. Now (18) and (20) imply

$$\frac{\partial^{|\beta|+3} h_j}{\partial x_i \partial x_k \partial x_p \partial x^\beta} = \frac{\partial^{|\beta|+3} h_k}{\partial x_i \partial x_j \partial x_p \partial x^\beta}, \tag{22}$$

for $1 \leqslant j, k, p \leqslant n$ and $1 \leqslant i \leqslant n-1$. The lemma follows from (21) and (22). □

Lemma 6. *Let $y \in \partial\Omega$. There exist $r > 0$ and a function v analytic in the open ball $B(y, r)$ of $\mathbb{R}^n$ such that*

$$\frac{\partial v}{\partial x_j}(x) = h_j(x), \quad \forall x \in B(y, r),\ j = 1, \dots, n.$$

Proof. By Remark 1 there exists an open ball $B(y, r)$ in $\mathbb{R}^n$ such that

$$h_j(x) = \sum_{\alpha \geqslant 0} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} h_j}{\partial x^\alpha}(y)(x - y)^\alpha$$

for $x \in B(y, r)$ and $j = 1, \dots, n$. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. We define $v_\alpha \in \mathbb{R}$ as follows: $v_0 \in \mathbb{R}$

$$v_{(\alpha_1,0,\dots,0)} = \frac{\partial^{\alpha_1-1}h_1}{\partial x_1^{\alpha_1-1}}(y) \quad \text{for } \alpha_1 \geqslant 1,$$

$$\vdots$$

$$v_{(0,\dots,0,\alpha_n)} = \frac{\partial^{\alpha_n-1}h_n}{\partial x_n^{\alpha_n-1}}(y) \quad \text{for } \alpha_n \geqslant 1,$$

and, when there exist α_j, $\alpha_k \geqslant 1$ for j and k such that $1 \leqslant j < k \leqslant n$

$$v_\alpha = \frac{\partial^{|\beta|+1}h_j}{\partial x_k \partial x^\beta}(y) = \frac{\partial^{|\beta|+1}h_k}{\partial x_j \partial x^\beta}(y),$$

where $\beta = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \dots, \alpha_{k-1}, \alpha_k - 1, \alpha_{k+1}, \dots, \alpha_n)$ with the obvious modifications when $j = 1$ or $k = n$ or $k = j + 1$ and where the last equality is given by Lemma 5. Let $\varepsilon_1 = (1, 0, \dots, 0), \dots, \varepsilon_n = (0, \dots, 0, 1)$. Since

$$\begin{aligned} h_j(x) &= \sum_{\alpha \geqslant 0} \frac{1}{\alpha!} v_{\alpha+\varepsilon_j}(x-y)^\alpha \\ &= \frac{\partial}{\partial x_j}\left(\sum_{\alpha \geqslant 0} \frac{1}{\alpha!} v_\alpha (x-y)^\alpha \right), \end{aligned}$$

for $x \in B(y,r)$ and $1 \leqslant j \leqslant n$, we define an analytic function v in $B(y,r)$ by

$$v(x) = \sum_{\alpha \geqslant 0} \frac{1}{\alpha!} v_\alpha (x-y)^\alpha, \quad x \in B(y,r),$$

such that

$$\frac{\partial v}{\partial x_j}(x) = h_j(x), \quad x \in B(y,r),\ j = 1, \dots, n. \qquad \square$$

Lemma 7. *There exist an open connected neighborhood U of $\partial\Omega$ and a function v analytic in U such that*

$$\frac{\partial v}{\partial x_j}(x) = h_j(x), \quad \forall x \in U,\ j = 1, \dots, n.$$

Proof. Since $\partial\Omega$ is compact, using Lemma 6, we can find $y_1, \dots, y_m \in \partial\Omega$, $r_1, \dots, r_m > 0$ and $v_1, \dots, v_m$ analytic in $B(y_1, r_1), \dots, B(y_m, r_m)$ respectively such that

$$\partial\Omega \subset \bigcup_{k=1}^{m} B(y_k, r_k) = U$$

and

$$\frac{\partial v_k}{\partial x_j}(x) = h_j(x), \quad \forall x \in B(y_k, r_k),\ k = 1, \dots, m \text{ and } j = 1, \dots, n.$$

Moreover U is connected since $\partial\Omega$ is connected. Suppose that $A_{il} = B(y_i, r_i) \cap B(y_l, r_l) \neq \emptyset$. We have

$$\frac{\partial v_i}{\partial x_j}(x) = \frac{\partial v_l}{\partial x_j}(x), \quad \forall x \in A_{il},\ j = 1, \dots, n,$$

hence $v_i = v_l + c_{il}$ on A_{il} for some constant c_{il}. Define

$$v = \begin{cases} v_i & \text{on } B(y_i, r_i), \\ v_l + c_{il} & \text{on } B(y_l, r_l). \end{cases}$$

Then v is analytic on $B(y_i, r_i) \cup B(y_l, r_l)$ and

$$\frac{\partial v}{\partial x_j}(x) = h_j(x), \quad \forall x \in B(y_i, r_i) \cup B(y_l, r_l),\ j = 1, \dots, n.$$

The lemma follows by repeating this procedure. $\square$

Lemma 8. *There exist an open connected neighborhood V of Ω and a function w analytic in V such that*

$$\frac{\partial w}{\partial x_j}(x) = h_j(x), \quad \forall x \in V, \ j = 1, \dots, n.$$

Proof. With the notations of Lemma 7 let $y \in U \cap \Omega$ and let $r = d(y, \partial\Omega)$. Since h_j is analytic in Ω,

$$h_j(x) = \sum_{\alpha \geqslant 0} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} h_j}{\partial x^\alpha}(y)(x-y)^\alpha$$

for $x \in B(y,r)$ and $j = 1, \dots, n$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we define $w_\alpha \in \mathbb{R}$ as follows: $w_0 = v(y)$

$$w_{(\alpha_1, 0, \dots, 0)} = \frac{\partial^{\alpha_1 - 1} h_1}{\partial x_1^{\alpha_1 - 1}}(y) \quad \text{for } \alpha_1 \geqslant 1,$$

$$\vdots$$

$$w_{(0, \dots, 0, \alpha_n)} = \frac{\partial^{\alpha_n - 1} h_n}{\partial x_n^{\alpha_n - 1}}(y) \quad \text{for } \alpha_n \geqslant 1,$$

and, when there exist $\alpha_j, \alpha_k \geqslant 1$ for j and k such that $1 \leqslant j < k \leqslant n$

$$w_\alpha = \frac{\partial^{|\beta|+1} h_j}{\partial x_k \partial x^\beta}(y) = \frac{\partial^{|\beta|+1} h_k}{\partial x_j \partial x^\beta}(y),$$

where $\beta = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \dots, \alpha_{k-1}, \alpha_k - 1, \alpha_{k+1}, \dots, \alpha_n)$ with the obvious modifications when $j = 1$ or $k = n$ or $k = j + 1$ and where the last equality is given by Lemma 7. Let again $\varepsilon_1 = (1, 0, \dots, 0), \dots, \varepsilon_n = (0, \dots, 0, 1)$. Since

$$\begin{aligned} h_j(x) &= \sum_{\alpha \geqslant 0} \frac{1}{\alpha!} w_{\alpha + \varepsilon_j}(x-y)^\alpha \\ &= \frac{\partial}{\partial x_j}\left(\sum_{\alpha \geqslant 0} \frac{1}{\alpha!} w_\alpha (x-y)^\alpha \right), \end{aligned}$$

for $x \in B(y,r)$ and $j = 1, \dots, n$, we define an analytic function w in $B(y,r)$ by

$$w(x) = \sum_{\alpha \geqslant 0} \frac{1}{\alpha!} w_\alpha (x-y)^\alpha, \quad x \in B(y,r),$$

such that

$$\frac{\partial w}{\partial x_j} = h_j \quad \text{on } U \cap B(y,r), \ j = 1, \dots, n.$$

Since

$$\frac{\partial v}{\partial x_j} = h_j \quad \text{on } U \cap B(y,r), \ j = 1, \dots, n,$$

we deduce that there exists a constant d such that $w = v + d$ on $U \cup B(y,r)$. Therefore we still have

$$\frac{\partial w}{\partial x_j} = h_j \quad \text{on } U \cup B(y,r), \ j = 1, \dots, n,$$

and the lemma follows easily. □

Lemma 9. *There exists a function u analytic in Ω such that*

$$\Delta u + 1 = 0 \quad \text{in } \Omega,$$

$$u = 0, \qquad \frac{\partial u}{\partial \nu} = d \quad \text{on } \partial\Omega,$$

where d denotes some constant.

Proof. The function w given by Lemma 8 satisfies

$$\Delta w = a \quad \text{in } \Omega$$

for some constant $a \in \mathbb{R}$ since

$$\frac{\partial \Delta w}{\partial x_j} = \Delta \frac{\partial w}{\partial x_j} = \Delta h_j = 0 \quad \text{in } \Omega,\ j = 1, \dots, n.$$

For $1 \leqslant k \leqslant n-1$ we have

$$\begin{aligned} \frac{\partial}{\partial x_k}(w) &= \frac{\partial w}{\partial x_k} + \frac{\partial w}{\partial x_n}\frac{\partial \varphi}{\partial x_k} \\ &= \frac{1}{\nu_n}(h_k \nu_n - h_n \nu_k) = 0 \end{aligned}$$

on $\partial\Omega$ by (1). Since $\partial\Omega$ is connected, there exists a constant $b \in \mathbb{R}$ such that $w = b$ on $\partial\Omega$. If $a = 0$, then $w = b$ in Ω a contradiction to the fact that $h_j \not\equiv 0$. Therefore $a \neq 0$. Now

$$u = -\frac{1}{a} w + \frac{b}{a}$$

satisfies Lemma 9. □

Proof of Theorem 1 completed. Theorem 1 follows from Lemma 9 and Theorem 2. □

3. Proof of the corollary

For every $j = 1, \dots, n$ let h_j be a harmonic function satisfying (1) with $c = C$. $C \neq 0$ by Remark 2. Let $h \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a harmonic function. From Green's identity we get

$$\int_{\partial\Omega} h_j \frac{\partial h}{\partial \nu}\, ds = \int_{\partial\Omega} h \frac{\partial h_j}{\partial \nu}\, ds \tag{23}$$

for $j = 1, \dots, n$. Then using (1), (3) and (23) we obtain

$$\int_{\partial\Omega} h\left(\frac{\partial h_j}{\partial \nu} + \big(1 + C(n-1)H\big)\nu_j\right) ds = 0 \tag{24}$$

for $j = 1, \dots, n$. Now choose h such that

$$h = \frac{\partial h_j}{\partial \nu} + \big(1 + C(n-1)H\big)\nu_j \quad \text{on } \partial\Omega.$$

Then (24) implies that (2) holds with $c = C$, and Theorem 1 completes the proof.

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